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On Twisted Quintic Curves.

By E. C. Colpitts.

INTRODUCTION.

The method of depicting on a conic the rational R_5 that lies on a hyperboloid is the same as that used by Emil Weyr for the rational quartic curve.* The theory of biquadratic involution used in this paper is taken from two other articles by Weyr.†

The following is a list of the papers that have been published on twisted quintic curves:

(1). On the rational curves.

E. Bertini. Su le curve gobbe razionali del 5° ordine. *Collect. math. in memoriam D. Chelini*, Mediolani, 1881, pp. 313–326.

L. Berzolari. Su la curva gobba razionale del quint'ordine. *Atti della R. Acc. dei Lincei*, Rome, 1894, pp. 305–341.

G. K. Nugteren. Rationale Ruimtekrommen van de vijfde orde. *Diss., Utrecht*, 1901, 67 pp.

Alfons Schmitz. Ueber eine bemerkenswerte Raumkurve fünfter Ordnung. *Diss., Munich*, 1887, 58 pp.

(2). On the curves of genus one.

Emil Weyr. Ueber Raumcurven fünfter Ordnung vom Geschlechte Eins. *Wiener Berichte*, vol. 90, 1884, pp. 206–225; vol. 92, 1885, pp. 498–523; vol. 97, 1888, pp. 592–617.

D. Montesano. Su la curva gobba di 5° ordine e di genere 1. *Acc. Napoli*, vol. 27, 1888, pp. 181–188.

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*Ueber die Abbildung einer rationalen Raumcurve vierter Ordnung auf einen Kegelschnitt. *Wiener Berichte*, vol. 72, 1875.

† Ueber biquadratische Involutionen erster Stufe. *Wiener Berichte*, vol. 83, 1881, pp. 300–320.

Ueber Ausartungen biquadratischer Involutionen. *Wiener Berichte*, vol. 83, pp. 807–827.

(3). On the curves of genus two.

H. E. Timerding. Ueber eine Raumcurve fünfter Ordnung. *Crelle*, vol. 123, 1901, pp. 284–311.

I have made use of the paper by Nugteren in the discussion of the rational quintic curves, and those by Weyr and Montesano for the curves of genus one.

I have treated the rational quintics by writing the equations in terms of a parameter. The conditions for singularities on the rational curve are found in this way. The (1, 4) quintic lying on a hyperboloid is, however, studied by depicting it on a conic. I have thus found twelve curves of this type besides the non-singular case. At the close of the chapter dealing with rational quintics is a list of the quintic curves that belong to a linear complex. The quintic curve of genus one on a cubic surface F_3 is depicted on a plane, and the number of times that each line of F_3 cuts R_5 is determined. The same thing is done for the case when F_3 has a double point. It is then shown that the trisecants of R_5 of genus one belong to a linear complex and that the pole of any plane lies on the conic determined by the five points in which the plane cuts R_5. By means of the method used by Clebsch I have written the equation of the plane projection of the quintic curve of genus two and found the maximum number of inflexional generators of the curve.

CHAPTER I.

ON THE RATIONAL QUINTIC CURVES.

§1. *Proof for the Existence of at Least one Quadrisecant.*

The rational quintic curves are divided into two classes; every quintic R_5^0 of the first kind is the partial intersection of two surfaces of the third order, the residual curve consisting of a curve R_3 and a line which does not cut this curve; or it may be the partial intersection of two ruled surfaces of the third order which have the same double directrix. The rational R_5^0 of the second kind is the partial intersection of a quadric surface F_2 with a ruled surface of the fourth order which has a quadrisecant S_4 of R_5^0 for a triple line; or it is the intersection of a quadric surface F_2 with a quartic surface F_4, the residual intersection being three skew lines.

Besides the non-singular rational quintic curves ($h = 6$), there are various other forms, viz.:

$h = 5,\ H = 1$; $h = 5,\ \beta = 1$; $h = 4,\ H = 2$; $h = 4,\ H = 1,\ \beta = 1$;

$h = 4$, $\beta = 2$; $h = 3$, with a triple point. A triple point may have three distinct tangents; or all three coincident.* For the Cayley numbers of the quintic I use the symbols and relations as given by E. Pascal, Repertorium der höheren Mathematik, vol. 2, pp. 225–6.

A rational quintic curve R_5 may be expressed by

$$\rho x = \sum_{i=0}^{5} a_i t^{5-i}, \quad \rho y = \sum_{i=0}^{5} b_i t^{5-i}, \quad \rho z = \sum_{i=0}^{5} c_i t^{5-i}, \quad \rho w = \sum_{i=0}^{5} d_i t^{5-i}.$$

If R_5 be cut by a plane $Ax + By + Cz + Dw = 0$ the parameters of the points of intersection are defined by $\sum_{i=0}^{5} (Aa_i + Bb_i + Cc_i + Dd_i) t^{5-i} = 0$, say t_1, t_2, t_3, t_4, t_5. If $\Sigma t_1 = \Sigma_1$, $\Sigma t_1 t_2 = \Sigma_2$, etc., we have

$$(Aa_0) \Sigma_i + (-1)^{i-1} (Aa_i) = 0, \; i = 1, \ldots, 5.$$

where $(Aa_i) = Aa_i + Bb_i + Cc_i + Dd_i$. Eliminate A, B, C, D between these equations. The result will be the conditions that the points $t_1 \ldots t_5$ are coplanar. After various reductions one condition may be written,

$$(a_0 b_1 c_2 d_3) \Sigma_4 + (a_0 b_1 c_2 d_4) \Sigma_3 + (a_0 b_1 c_3 d_4) \Sigma_2 + (a_0 b_2 c_3 d_4) \Sigma_1 + (a_1 b_2 c_3 d_4) = 0$$

or say,

$$\text{I.} \qquad P_1 \Sigma_4 + P_2 \Sigma_3 + P_3 \Sigma_2 + P_4 \Sigma_1 + P_5 = 0.$$

The remaining condition is

$$(a_0 b_1 c_2 d_3) \Sigma_5 - (a_0 b_1 c_2 d_5) \Sigma_3 - (a_0 b_1 c_3 d_5) \Sigma_2 - (a_0 b_2 c_3 d_5) \Sigma_1 - (a_1 b_2 c_3 d_5) = 0$$

or

$$\text{I}'. \qquad q_0 \Sigma_5 + q_2 \Sigma_3 + q_3 \Sigma_2 + q_4 \Sigma_1 + q_5 = 0.$$

If definite values are given to t_1, t_2, t_3, t_4, then t_5 must satisfy the equations.

$$(P_1 \sigma_4 + P_2 \sigma_3 + P_3 \sigma_2 + P_4 \sigma_1 + P_5) + (P_1 \sigma_3 + P_2 \sigma_2 + P_3 \sigma_1 + P_4) t_5 = 0.$$
$$(q_2 \sigma_3 + q_3 \sigma_2 + q_4 \sigma_1 + q_5) + (q_0 \sigma_4 + q_2 \sigma_2 + q_3 \sigma_1 + q_4) t_5 = 0.$$

where σ_i denotes the fundamental symmetric functions of t_1, t_2, t_3, t_4 of weight i. The value of t_5 is indeterminate if

$$\begin{aligned} P_1 \sigma_4 + P_2 \sigma_3 + P_3 \sigma_2 + P_4 \sigma_1 + P_5 &= 0. \\ P_1 \sigma_3 + P_2 \sigma_2 + P_3 \sigma_1 + P_4 &= 0. \\ q_2 \sigma_3 + q_3 \sigma_2 + q_4 \sigma_1 + q_5 &= 0. \\ q_0 \sigma_4 + q_2 \sigma_2 + q_3 \sigma_1 + q_4 &= 0. \end{aligned}$$

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* Two coincident tangents can also exist, *e. g.*, $x = t^4 (t-1)$, $y = t^3 (t-1)^2$, $z = t^2 (t-1)^3$. $w = \phi_5 (t)$. (See Van der Vries: *Proc. Amer. Acad.*, vol. 38, p. 524, footnote).

These four equations are linear in t_1, t_2, t_3, t_4. Hence *every rational R_5 has one quadrisecant.* R_5 has ∞ quadrisecants if these equations are not independent, that is, if

$$\text{II.} \qquad \begin{vmatrix} P_1, & P_2, & P_3, & P_4 \\ & P_1, & P_2, & P_3 \\ & q_2, & q_3, & q_4 \\ q_0, & & q_2, & q_3 \end{vmatrix} = 0.$$

Synthetic proof: Establish a projectivity between the points of R_5^0 and any S_3, having the three points of intersection as corresponding points. Draw the lines connecting a point x on S_3 with its corresponding point x' on R_5. These lines will generate a ruled surface on which s_3 is a simple directrix. Every plane through S_3 cuts R_5 in two points x' not on the line, hence contains two generators. The line S_3 cannot be a generator since each point of R_5^0 upon it is self-correspondent, hence R_5^0 lies on an infinite number of cubic scrolls. One point x' lies on each generator, hence a plane through the double directrix will only cut the curve in one variable point, from which it follows that R_5^0 has a quadrisecant. If it has only one, every such ruled cubic has this same quadrisecant for double directrix: if it has more than one it lies on a hyperboloid.*

§ 2. R_5 *as the Intersection of Two Ruled Surfaces of the Third Order.*†

From the preceding theorem it is seen that every R_5^0 can be defined as the intersection of two ruled cubic surfaces, having a double line in common. The equations

$$U_3 + zU_2 + V_2 = 0, \quad U_3' + zU_2' + V_2' = 0,$$

in which U_i, V_2 are polynomials in x, y of degree i, 2, represent two ruled surfaces each having the line $x = 0$, $y = 0$, for double directrix. Their intersection is made up of the double directrix d_2 counted four times and a twisted quintic curve R_5 having d_2 for a quadrisecant.

Let the surface

$$(a_0 x^3 + 3a_1 x^2 y + 3a_2 xy^2 + a_3 y^3) + z(b_0 x^2 + 2b_1 xy + b_2 y^2) + (c_0 x^2 + 2c_1 xy + c_2 y^2) = 0.$$

be given. Any plane through the double directrix is $y = mx$. Substitute $y = mx$

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* Another proof that every rational R_5 has one quadrisecant is given by Bertini in *Collectanea Mathematica in Memoriam D. Chelini*, Mediolani, 1881.

† G. K. Nugteren, Rationale Ruimte krommen van de Vijfde Orde, Diss., Utrecht, 1901.

in the above equation. The resulting equation is

$$(a_0 x^3 + 3a_1 mx^3 + 3a_2 m^2 x^3 + a_3 m^3 x^3) + z(b_0 x^2 + 2b_1 mx^2 + b_2 m^2 x^2) \\ + (c_0 x^2 + 2c_1 mx^2 + c_2 m^2 x^2) = 0.$$

Besides $x^2 = 0$, we have

$$(a_0 x + 3a_1 mx + 3a_2 m^2 x + a_3 m^3 x) + z(b_0 + 2b_1 m + b_2 m^2) \\ + (c_0 + 2c_1 m + c_2 m^2) = 0$$

or

$$a_m^3 x + b_m^2 z + c_m^2 = 0.$$

The intersection of $y = mx$ with $x^2 = 0$ defines d_2; $y = mx$ and $a_m^3 x + b_m^2 z + c_m^2 = 0$ define a generator of the surface. We have the (1, 2) relation between z and m

$$z(b_0 + 2b_1 m + b_2 m^2) + (c_0 + 2c_1 m + c_2 m^2) = 0$$

when the planes $x = 0$, and $a_m^3 x + b_m^2 z + c_m^2 = 0$ intersect. Hence, through every point of d_2 pass two planes cutting out generators of F_3, or two generators pass through every point of d_2.

For $m = 0$, $z = -\frac{c_0}{b_0}$. The other value of m corresponding to $z = -\frac{c_0}{b_0}$ is $m = -\frac{2(b_0 c_1)}{(b_0 c_2)}$. The general equation of a plane through these two generators is $Ax + By + Cz + D = 0$. This plane contains two lines which are the intersections of the planes $y = mx$ and $a_m^3 x + b_m^2 z + c_m^2 = 0$, for $m = 0$ and $m = -\frac{2(b_0 c_1)}{(b_0 c_2)}$ or say, $= -2t$. If $m = 0$, we have $y = mx = 0$, and $a_0 x + b_0 z + c_0 = 0$.

Therefore $Ax + By + Cz + D = 0$, must be the same plane as $a_0 x + b_0 z + c_0 + B'y = 0$ (1) for some quantity B'. Similarly if $m = -2t$, we have the two planes $y + 2tx = 0$, and

$$(a_0 - 6a_1 t + 12a_2 t^2 - 8a_3 t^3)x + (b_0 - 4b_1 t + b_2 t^2)z + (c_0 - 4c_1 t + 4c_2 t^2) = 0.$$

As before, the plane $Ax + By + Cz + D = 0$ must coincide with the plane

$$(a_0 - 6a_1 t + 12a_2 t^2 - 8a_3 t^3)x + (b_0 - 4b_1 t + b_2 t^2)z \\ + (c_0 - 4c_1 t + 4c_2 t^2) + B''(y + 2tx) = 0. \qquad (2)$$

for some value B''. Since (1) and (2) represent the same plane.

$$\frac{a_0 - 6a_1 t + 12a_2 t^2 - 8a_3 t^3 + 2B' t}{a_0} = \frac{B''}{B'} = \frac{b_0 - 4b_1 t + 4b_2 t^2}{b_0} \\ = \frac{c_0 - 4c_1 t + 4c_2 t^2}{c_0}.$$

From these relations we find that

$$B' = \frac{c_0\,(3a_1 b_0 - 2a_0 b_1 - 6a_2 b_0 t + 2a_0 b_2 t + 4a_3 t^2)}{b_0 \qquad c_0 - 4c_1 t + 4c_2 t^2}.$$

If now we substitute for t its value $\frac{(b_0 c_1)}{(b_0 c_2)}$, we find

$$B' = \frac{3a_1 (b_0 c_2)^2 - 6a_2 (b_0 c_2)\,(b_0 c_1) - 2a_0 (b_0 c_2)\,(b_1 c_2) + 4a_3 (b_0 c_1)^2}{(b_0 c_2)^2 - 4\,(b_0 c_1)\,(b_1 c_2)}.$$

Again, for $m = \infty$, $z = -\frac{c_2}{b_2}$. The other value of m is found to be $-\frac{1}{2}\,\frac{(b_0 c_2)}{(b_1 c_2)}$. A general plane through the two generators corresponding to $m = \infty$ and $m = -\frac{1}{2}\,\frac{(b_0 c_2)}{(b_1 c_2)} = -\frac{1}{2}\,v$, say, is $Ax + By + Cz + D = 0$. For $m = \infty$,

$$a_3 y + b_2 z + c_2 + A' x = 0 \tag{3}$$

represents the same plane. For $m = -\frac{v}{2}$, we have the two planes $y + \frac{v}{2}\,x = 0$,

$$\left(a_0 - \frac{3}{2} a_1 v + \frac{3}{4} a_2 v^2 - \frac{a_3}{8} v^3\right) x + \left(b_0 - b_1 v + \frac{b_2}{4} v^2\right) z + \left(c_0 - c_1 v + \frac{c_2}{4} v^2\right) = 0.$$

Any plane through their line of intersection is

$$\left(a_0 - \frac{3}{2} a_1 v + \frac{3}{4} a_2 v^2 - \frac{a_3}{8} v^3\right) x + \left(b_0 - b_1 v + \frac{b_2}{4} v^2\right) z$$
$$+ \left(c_0 - c_1 v + \frac{c_2}{4} v^2\right) + B''\left(y + \frac{v}{2} x\right) = 0. \tag{4}$$

If the planes (3) and (4) are identical

$$\frac{a_0 - \frac{3}{2} a_1 v + \frac{3}{4} a_2 v^2 - \frac{a_3}{8} v^3 + \frac{B'' v}{2}}{A'} = \frac{B''}{a_3} = \frac{b_0 - b_1 v + \frac{b_2}{4} v^2}{b_2}$$
$$= \frac{c_0 - c_1 v + \frac{c_2}{4} v^2}{c_2}.$$

It follows that

$$A' = c_2 \frac{\left(a_0 - \frac{3}{2} a_1 v + \frac{3}{4} a_2 v^2 + \frac{1}{2} a_3 \frac{c_0 v}{c_2} - \frac{1}{2} a_3 \frac{c_1}{c_2} v^2\right)}{c_0 - c_1 v + \frac{c_2}{4} v^2}$$
$$= \frac{3a_2 (b_0 c_2)^2 - 6a_1 (b_0 c_2)\,(b_1 c_2) - 2a_3 (b_0 c_1)\,(b_0 c_2) + 4a_0 (b_1 c_2)^2}{(b_0 c_2)^2 - 4\,(b_1 c_2)\,(b_0 c_1)}.$$

The intersection of the two planes

$$a_0 x + By + b_0 z + c_0 = 0$$
$$A'x + a_3 y + b_2 z + c_2 = 0$$

is the simple directrix d_1 of F_3. The coordinates of the point of intersection of d_1 with the xy plane are $z = 0$,

$$x = \frac{-a_3 c_0 + c_2 B}{a_0 a_3 - A'B}, \quad y = \frac{-a_0 c_2 + c_0 A'}{a_0 a_3 - A'B}.$$

Let $B = \frac{t_5}{n_4}$, $A' = \frac{t'_5}{n_4}$. Then $x = \frac{-a_3 c_0 n_4 + c_2 t_5}{a_0 a_3 n_4^2 - t_5 t'_5} n_4 = \frac{N}{D} n_4$.

$$N = -a_3 c_0 [(b_0 c_2)^2 - 4(b_0 c_1)(b_1 c_2)] + c_2 [3a_1 (b_0 c_2)^2$$
$$- 6a_2 (b_0 c_2)(b_0 c_1) - 2a_0 (b_0 c_2)(b_1 c_2) + 4a_3 (b_0 c_1)^2].$$

But N is divisible by $(b_0 c_2)$. For we have

$$\begin{aligned} &4a_3 (b_0 c_1) [(b_0 c_1) c_2 + (b_1 c_2) c_0] \\ &\quad = 4a_3 (b_0 c_1) [b_0 c_1 c_2 - b_1 c_0 c_2 + b_1 c_2 c_0 - b_2 c_1 c_0] \\ &\quad = 4a_3 (b_0 c_1)(b_0 c_2) c_1. \end{aligned}$$

$$\begin{aligned} D &= a_0 a_3 [(b_0 c_2)^2 - 4(b_1 c_2)(b_0 c_1)]^2 - [(P(b_0 c_2) + 4a_3 (b_0 c_1)^2) \\ &\qquad \times (Q(b_0 c_2) + 4a_0 (b_1 c_2)^2)] \\ &= a_0 a_3 [(b_0 c_2)^4 - 8(b_0 c_2)^2 (b_1 c_2)(b_0 c_1) + 16 (b_1 c_2)^2 (b_0 c_1)^2] \\ &\quad - [R(b_0 c_2) + 16 a_0 a_3 (b_1 c_2)^2 (b_0 c_1)^2] \\ &= a_0 a_3 [(b_0 c_2)^4 - 8(b_0 c_2)^2 (b_1 c_2)(b_0 c_1)] - R(b_0 c_2). \end{aligned}$$

Therefore, both N and D are divisible by $(b_0 c_2)$, and the numerator and the denominator of x are both of degree eight. Similarly it may be shown the numerator and denominator of y are also of degree eight.

Form the pencil of surfaces

$$(U_3 + zU_2 + V_2) + t(U'_3 + zU'_2 + V'_2) = 0,$$

all of which pass through R_5. Then for a we write $a + ta'$; for b, $b + tb'$; for c, $c + tc'$. The simple directrix meets the xy plane in the point $x = \frac{f_8(t)}{h_8(t)}$, $y = \frac{g_8(t)}{h_8(t)}$, $z = 0$. Hence the locus of the point in which d_1 cuts the xy plane is a rational curve of order eight. Since the double directrix d_2 has four points on R_5, each generator has only one point. Hence, d_1 is a trisecant of R_5. *The simple directrices of the surfaces of the pencil $F_3 + tF'_3$ are the trisecants of R_5; and the trisecants form a rational ruled surface of order eight.* Hence, R_5 lies on ∞^1

ruled cubic surfaces all of which have the line d_2 for double directrix, and each surface has a different trisecant of R_5 for simple directrix.

We will now find the number of Cayley cubic scrolls in this pencil of surfaces. The equation of the Cayley scroll can be put in the form

$$U_3 + (Px + qy)\,[(b_0 x + b_1 y)z + (c_0 x + c_1 y)] = 0.$$

If

$$U_3 + (b_0 x^2 + 2b_1 xy + b_2 y^2)z + (c_0 x^2 + 2c_1 xy + c_2 y^2) = 0.$$

and

$$U_3' + (b_0' x^2 + 2b_1' xy + b_2' y^2)z + (c_0' x^2 + 2c_1' xy + c_2' y^2) = 0.$$

are the equations of two Cayley scrolls, then the terms $U_2 + tU_2'$ and $V_2 + tV_2'$ have a common factor, when we form the pencil $F_3 + tF_3' = 0$; or

$$(b_0 + tb_0')\left(\frac{x}{y}\right)^2 + 2\,(b_1 + tb_1')\left(\frac{x}{y}\right) + (b_2 + tb_2') = 0$$

$$(c_0 + tc_0')\left(\frac{x}{y}\right)^2 + 2\,(c_1 + tc_1')\left(\frac{x}{y}\right) + (c_2 + tc_2') = 0$$

have a root $\frac{x}{y}$ in common. The resultant of these two equations is of order four in the coefficients. Hence, *R_5 lies on four Cayley scrolls.*

The number of Cayley scrolls can be found by another method. R_5 has only one quadrisecant. Take three of the points of S_4 on R_5 as corresponding to the same three points of R_5. This fixes the projectivity between the points of S_4 and of R_5. Any plane through S_4 cuts R_5 in one point. The lines joining such points to their corresponding points on S_4 generate a Cayley cubic scroll, for which S_4 is the directrix. The three coincidences can be chosen among the four points in four different ways. Hence, there are four Cayley scrolls on which R_5 lies.

It is evident that the following statement in Pascal's Repertorium der höheren Mathematik, Vol. 2, p. 266, must be incorrect; "Jede R_5^0 der ersten Species ist der theilweise Schnitt zweier Regelflächen 3^{ter} Ordnung, welche ausserdem eine Doppelgerade und zwei andere Gerade gemeinschaftlich haben, von denen die eine die Doppelgerade schneidet und die andere nicht."

§ 3. *Plane projection of R_5.*

It has been shown that the trisecants form a rational ruled surface F_8 of order eight. Since R_5 projects from any one of its points into a rational plane curve of order four, it follows that three trisecants can be drawn from every point of R_5. Hence R_5 is a triple curve on F_8. No other trisecants can be drawn through the four points in which S_4 cuts R_5; for if this were possible a plane through S_4 and S_3 would meet R_5 in six points. Hence, S_4 counts for six generators. The complete nodal curve of F_8 consists of the original R_5 counted three times and S_4 counted as six double generators.

From any point not on the curve, R_5 projects into a rational plane quintic C_5. The configuration of cusps and double points on C_5 depends on the position in space of the point of projection. From any point on S_4, R_5 projects into a C_5 with a fourfold point; from any point of F_8 into a C_5 with one triple point and three double points. The tangents of R_5 form a planar developable D_8 of order eight. The point of tangency projects into a cuspidal point. Hence, from an ordinary point an D_8, R_5 projects into a C_5 with one cusp and five double points; from a point on the double curve of D_8 into a C_5 with two cusps and four double points; from one of the triple points of D_8 into a C_5 having three cusps and three double points. The plane projection of R_5 from an arbitrary point in space has six double points.

§ 4. *Derivation of $F(t)$ and the Conditions for some Singularities.*

If from equations I, I′ we eliminate t_5, and let the remaining parameters all become equal we have.

$$\begin{aligned}F(t) = {} & P_1 q_0 t^8 + 4P_2 q_0 t^7 + (6P_3 q_0 - 10P_1 q_2) t^6 + (4P_4 q_0 - 20P_1 q_3) t^5 \\ & + (P_5 q_0 + 20P_3 q_2 - 20P_2 q_3 - 15P_1 q_4) t^4 + (20P_4 q_2 - 20P_2 q_4 - 4P_1 q_5) t^3 \\ & + (6P_5 q_2 + 10P_4 q_3 - 10P_3 q_4 - 6P_2 q_5) t^2 + 4(P_5 q_3 - P_3 q_5) t \\ & + (P_5 q_4 - P_4 q_5) = 0.\end{aligned}$$

The roots of $F(t)$ are the parameters of the points of contact of the stationary planes of R_5. Hence, *R_5 has eight stationary planes.*

Writing S_1 for $t_1 + t_2 + t_3$, S_2 for $t_1 t_2 +$ etc.; $S_3 = t_1 t_2 t_3$, the conditions that five points should be coplanar may be written

$$\begin{aligned}P_1 [S_3 (t_4 + t_5) + S_2 t_4 t_5] + P_2 [S_3 + S_2 (t_4 + t_5) + S_1 t_4 t_5] \\ + P_3 [S_2 + S_1 (t_4 + t_5) + t_4 t_5] + P_4 [S_1 + (t_4 + t_5)] + P_5 = 0.\end{aligned}$$

$$\begin{aligned}q_0 S_3 t_4 t_5 + q_2 [S_3 + S_2 (t_4 + t_5) + S_1 t_4 t_5] + q_3 [S_2 + S_1 (t_4 + t_5) + t_4 t_5] \\ + q_4 [S_1 + (t_4 + t_5)] + q_5 = 0.\end{aligned}$$

If t_1, t_2, t_3 are collinear they lie in a plane no matter what values are given to $(t_4 + t_5)$ or to $t_4 t_5$. Hence

$$\text{III.}\quad \frac{P_2 S_3 + P_3 S_2 + P_4 S_1 + P_5}{q_2 S_3 + q_3 S_2 + q_4 S_1 + q_5} = \frac{P_1 S_3 + P_2 S_2 + P_3 S_1 + P_4}{q_2 S_2 + q_3 S_1 + q_4} = \frac{P_1 S_2 + P_2 S_1 + P_3}{q_0 S_3 + q_2 S_1 + q_3}.$$

express the condition that three points of R_5 lie on a trisecant.

Let t_3 be given the fixed value t, and write T_1 for $t_1 + t_2$ and T_2 for $t_1 t_2$. These equations may then be written

$$\frac{(P_5 + P_4 T_1 + P_3 T_2) + (P_4 + P_3 T_1 + P_2 T_2) t}{(q_5 + q_4 T_1 + q_3 T_2) + (q_4 + q_3 T_1 + q_2 T_2) t} = \frac{(P_4 + P_3 T_1 + P_2 T_2) + (P_3 + P_2 T_1 + P_1 T_2) t}{(q_4 + q_3 T_1 + q_2 T_2) + (q_3 + q_2 T_1) t} = \frac{(P_3 + P_2 T_1 + P_1 T_2) + (P_2 + P_1 T_1) t}{(q_3 + q_2 T_1) + (q_2 + q_0 T_2) t}.$$

Put each of these fractions equal to K and eliminate T_1, T_2, 1, thus:

$$\begin{vmatrix} (P_5 + P_4 t) - K(q_5 + q_4 t), & (P_4 + P_3 t) - K(q_4 + q_3 t), & (P_3 + P_2 t) - K(q_3 + q_2 t) \\ \dots\dots\dots\dots, & \dots\dots\dots\dots, & \dots\dots\dots\dots \\ (P_3 + P_2 t) - K(q_3 + q_2 t), & \dots\dots\dots\dots, & \dots\dots\dots\dots \end{vmatrix} = 0.$$

This is a cubic equation in K; hence K has three values K_1, K_2, K_3. Put two of these equations each equal to K_i. This determines T_1 and T_2, and therefore, one pair of roots $t_1 t_2$ on a line with t. Hence: *Three trisecants can be drawn through every point of* R_5.

If $t_1 = t_2 = t_3$ the trisecant becomes a tangent at the point of inflexion. Hence, the condition for a linear inflexion may be found by putting $t_1 = t_2 = t_3$ in III. The resulting equations are

$$\frac{P_2 t^3 + 3P_3 t^2 + 3P_4 t + P_5}{q_2 t^3 + 3q_3 t^2 + 3q_4 t + q_5} = \frac{P_1 t^3 + 3P_2 t^2 + 3P_3 t + P_4}{3q_2 t^2 + 3q_3 t + q_4} = \frac{3P_1 t^2 + 3P_2 t + P_3}{q_0 t^3 + 3q_2 t + q_3}.$$

This gives only two independent relations which may be written $\sum_{i=0}^{6} V_i t^i = 0$, $\sum_{i=0}^{6} S_i t^i = 0$, where the coefficients V_i, S_i are of order eight in the coefficients a, b, c, d. If R_5 have a linear inflexion these two sextics must have a common root.

The resultant IV of degree ninety-six expresses the condition for a linear inflexion on R_5.

When two stationary planes become coincident, $F(t) = 0$ must have a double root. The coefficients of $F(t)$ are of order eight in a, b, c, d; hence its discriminant V is of order one hundred and twelve.

When R_5 has an inflexion, two stationary planes coincide. Hence IV is a factor of V.

A plane having five point contact with R_5 counts for two stationary planes, that is, the parameter of the point of contact must be a double root of $F(t)$. To find the condition for such a plane, put $t_1 = t_2 = t_3 = t_4 = t_5$ in equations I and I′. The resulting equations are

$$5P_1 t^4 + 10P_2 t^3 + 10P_3 t^2 + 5P_4 t + P_5 = 0.$$
$$q_0 t_5 + 10q_2 t^3 + 10q_3 t^2 + 5q_4 t + q_5 = 0.$$

Their resultant VI is of degree thirty-six in a, b, c, d. It follows that VI is a factor of V.

When R_5 has an actual double point, two different values of the parameter correspond to the double point. Let t and u be these two values and let ρ and σ be the two factors of proportionality. Hence we have

$$\rho x = \sum_{i=0}^{5} a_i t^{5-i}; \quad \rho y = \sum_{i=0}^{5} b_i t^{5-i}; \quad \rho z = \sum_{i=0}^{5} c_i t^{5-i}, \quad \rho w = \sum_{i=0.}^{5} d_i t^{5-i};$$

and

$$\sigma x = \sum_{i=0.}^{5} a_i u^{5-i}; \quad \sigma y = \sum_{i=0.}^{5} b_i u^{5-i}; \quad \sigma z = \sum_{i=0}^{5} c_i u^{5-i}; \quad \sigma w = \sum_{i=0}^{5} d_i u^{5-i}.$$

Multiplying the first equations by σ and the second by ρ and subtracting we get four equations of the form

$$\sum_{i=0}^{5} a_i (\sigma t^{5-i} - \rho u^{5-i}) = 0 \text{ for } a, b, c, d.$$

In order to eliminate t, u, and $\frac{\rho}{\sigma}$, we form the four following identical equations

$$x(\sigma t^i - \rho u^i) + y(\sigma t^{i+1} - \rho u^{i+1}) + z(\sigma t^{i+2} - \rho u^{i+2}) = 0 \qquad (i = 0, 1, 2, 3).$$

where $\frac{x}{z} = tu, \frac{y}{z} = -(t + u)$. Take any two of these equations as the first and second and eliminate the expressions $(\sigma t^i - \rho u^i)$, $i = 0, 1, - 5$.

This gives the determinant

$$\begin{vmatrix} a_5, & a_4, & a_3, & a_2, & a_1, & a_0, \\ b_5, & b_4, & b_3, & b_2, & b_1, & b_0, \\ c_5, & c_4, & c_3, & c_2, & c_1, & c_0, \\ d_5, & d_4, & d_3, & d_2, & d_1, & d_0, \\ x, & y, & z, & 0, & 0, & 0, \\ 0, & x, & y, & z, & 0, & 0, \end{vmatrix} = 0.$$

There are six terms in this equation, and five other similar equations in x, y, z. Hence, there are enough equations to eliminate $x^2, xy, y^2, xz, zy, z^2$. Denoting by (i, k) the determinant formed by striking out the $(5-i)$ *th* and $(5-k)$ *th* columns of the constants, we find the condition VII for a double point to be

$$\begin{vmatrix} (01), & (02), & (12), & (03)-(12), & (13), & (23) \\ (02), & (03)+(12), & (13), & (04), & (14)+(23), & (24) \\ (12), & (13), & (23), & (14)-(23), & (24), & (34) \\ (03), & (04)+(13), & (14), & (23)+(05), & (24)+(15), & (25) \\ (13), & (14)+(23), & (24), & (15), & (25)-(34), & (35) \\ (23), & (24), & (34), & (25)-(34), & (35), & (45) \end{vmatrix} = 0.$$

This condition is of degree twenty-four.

To find the condition for a cusp replace one of the equations in x, y, z, by the identity $x + ty + t^2 z = 0$, and eliminate x^2, etc., as before. The roots t and u are given by the equation $t_0 t^2 + t_1 t + t_2 = 0$ where t_0, t_1, t_2 are of degree twenty in a_i, b_i, c_i, d_i. But at a cusp $t = u$. Hence, the condition for a cusp VIII, is of order forty. In order that R_5 may have a cusp, VII and VIII must both be satisfied,* and $F_8(t)$ has a triple root.

The equation determining the parameters of the points on the quadrisecant can be easily found by means of the equations on page 311. Solving for $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ we find the equation determining the parameters to be of the form $Ut^4 - U_1 t^3 + U_2 t^2 - U_3 t + U_4 = 0$, where U and U_i are of degree sixteen. When a biquadratic equation has a pair of double roots $G = 0$, and $a^2 I - 1\ 2H^2 = 0$†. Hence, denoting these two equations by IX and IX$'$, we have the result that if R_5 has a bitangent, equations IX and IX$'$ of degrees forty-eight and sixty-four respectively must be satisfied. The conditions for four coincident roots gives three equations X, X$'$, X$''$, of orders thirty-two, thirty-two and forty-eight respectively that must be satisfied when R_5 has a four-point contact tangent. $F_8(t)$ has a four-fold root.

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* This discussion is taken from a paper by A. Brill, "Über die Doppelpunkte von Curven im Raume deren Geschlecht Null ist." *Math. Annalen*, 1871.

† These symbols H, I, G invariants of a binary quartic, are used by Burnside & Panton, Theory of Equations.

§ 5. *Enumeration of Possible Singularities and the Multiple Roots of $F(t)$.*

The non-singular quintic lying on a hyperboloid will be discussed later by another method. We will now consider the various possible singularities on the other rational quintics. Since the curve has only one quadrisecant, there may be one bitangent if conditions IX and IX$'$ are satisfied, or one four-point contact tangent when conditions X, X$'$, X$''$, are satisfied and $F(t)$ has a four-fold root. When R_5 has a linear inflexion conditions IV and V must be satisfied. For each inflexion $F(t)$ has a double root. If a plane has five-point contact with R_5 conditions V and VI must hold. Each such plane requires that $F(t)$ should have a double root. For a cusp, conditions VII and VIII must be satisfied and $F(t)$ has a triple root. There may be an inflexion at a plane having five-point contact when conditions IV and VI must be satisfied and $F(t)$ has a triple root. A cusp at a point whose osculating plane has five-point contact requires conditions VI, VII, and VIII to be satisfied; $F(t)$ has a four-fold root. A cusp at which the tangent has four-point contact with the curve gives rise to a five-fold root of $F(t)$; and a triple point with all three tangents coincident give rise to a six-fold root.

§ 6. *The Quintic with two Inflexions.*

The equation of the quintic with two inflexions may be written in the form,

$$x = t^4(t+a), \quad y = t^3(t+b), \quad z = t+c, \quad w = t(t+d),$$

in which a, b, c, d, are arbitrary.

Equations I and I$'$ become

$$\Sigma_2 + b\Sigma_1 + ab = 0, \text{ and } cd\Sigma_3 + c\Sigma_4 + \Sigma_5 = 0;$$

the condition that R_5 lies on a hyperboloid is, $d - c = 0$;

$$F(t) = t^2\,[t^4 + \tfrac{2}{3}(b+5c)\,t^3 + \tfrac{1}{6}(ab + 15bc + 20cd)\,t^2 \\ + \tfrac{2}{3}\,bc\,(a+5d)\,t + abcd] = 0.$$

$F(t)$ is of degree six since ∞ was taken as one point of inflexion.

The equation determining the parameters of the points on S_4 is

$$t^4 + b\,t^3 + (b^2 - ab)\,t^2 + \frac{bcd\,(a-b)}{d-c}\,t + \frac{bcd^2\,(a-b)}{d-c} = 0.$$

In the case of a four-point inflexion this equation is of the form $(t-x)^4=0$. This gives three possible cases.

(*a*) $x=0$, $b=0$. $F(t)=0$ reduces to $t^4(3t^2+10ct+10cd)=0$. The two remaining roots are equal if $5c=6d$. This curve has linear inflexions at ∞ and $-\frac{5}{3}c$, and a four-point contact tangent at 0. It belongs to the linear complex $P_{13}+2P_{42}=0$.

(*b*) $x=-\dfrac{b}{4}$, $8a=5b$, $b=16d$. Hence

$$F(t)=t^2\left(t+\frac{b}{4}\right)^4=0\,;\ a=\frac{5b}{8},\ b=b,\ c=\frac{b}{10},\ \text{and } d=\frac{b}{16}.$$

It belongs to the linear complex $P_{13}+2P_{42}=0$. (*a*) and (*b*) are the same curves; in (*a*), $t=0$ is a four-point inflexion; in (*b*), it is simply a linear inflexion.

(*c*) $x=-\dfrac{b}{4}$, $8a=5b$, $d=0$. Hence $b=0$, $a=0$, and

$$F(t)=t^6+\frac{10}{3}ct^5=0.$$

This curve has for its equations $x=t^5$, $y=t^4$, $z=t+c$, $w=t^2$. Hence, $t=0$ is a cusp at which the tangent has four-point contact with the curve. It is evident geometrically that the curve cannot belong to a linear complex. It was proved by Picard* that the necessary and sufficient condition that a unicursal twisted curve should belong to a linear complex is that the tangent at the point of contact of every stationary plane should have at least three-point contact with the curve, when the point is not a multiple point.

The conditions that the quadrisecant should become a bitangent are $bd=ca$, and $3b-4a=8d$.

If $F_8(t)$ is a square we have one of the following cases: $b=0$, $c=0$, $ab-25cd=0$, $ac-bd=0$.

If $b=0$, $F(t)=3t^6+10ct^5+10cdt^4=0$.

This curve has been already discussed.

If $c=0$, R_5 is composite.

If $ab=25cd$ we have the following cases:

(*d*) $ab-25cd=0$, $2b^2-25bc+45cd+50c^2=0$.
(*e*) $ab-25cd=0$, $b+5c=0$, $a+5d=0$, $d=-5c$.
(*f*) $ab-25cd=0$, $b+5c=0$, $a+5d=0$, $5c=7d$.

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* Applications de la théorie des complexes linéaires à l'étude des surfaces et des courbes gauches. *Annales de l'Ecole Normale*, 1877.

(*d*) The equations of the curve cannot be simplified by means of the relations.

(*e*) The curve does not belong to a linear complex. $F(t) = t^6 - 50c^2 t^4 + 625c^4 t^2 = 0$.

The four points of contact of the stationary planes are $-5c$, 0, $5c$ and ∞, and are therefore harmonic.

(*f*) This curve belongs to the linear complex

$$P_{13} + 2P_{42} = 0. \quad F(t) = t^2\left[t^4 - \frac{50}{7}c^2 t^2 + \frac{625}{49}c^4\right] = 0.$$

Since the curve belongs to a linear complex and has the four double roots $\frac{-5\sqrt{7}}{7}$, 0, $\frac{5\sqrt{7}}{7}$, ∞, it is possible for a rational quintic curve, not lying on a hyperboloid, to have four linear inflexions, and the four points of inflexion are harmonic. Berzolari in *Atti della Reale Accademia dei Lincei*, 1894, p. 330, in his paper, Su la curva gobba razionale del quint'ordine., fails to notice that when the points of contact of the stationary planes coincide in pairs, there may be four inflexions.

If $ac = bd$, there is only one new type of curve

$$(g) \quad 2b^2 - 25bc + 50c^2 + 36bd - 3ab - 60cd = 0.$$

Combining this with $ac = bd$, and $3b - 4a = 8d$, the conditions for a bitangent, we find $a = \frac{b}{2}$, $c = \frac{b}{4}$, $d = \frac{b}{8}$. This curve belongs to the complex $P_{13} + 2P_{42} = 0$.

$$F(t) = t^2\left[t^4 + \frac{3}{2}bt^3 + \frac{13}{16}b^2 t^2 + \frac{3}{16}b^3 t + \frac{b^4}{64}\right] = 0.$$

Hence the quintic curve can have four linear inflexions and one bitangent.

§ 7. *Some possible types of R_5.*

A quintic with three planes having five-point contact cannot have a linear inflexion. These three planes meet in a point and each counts for three osculating planes. But if the curve has one inflexion the class is eight. Hence, this type of R_5 cannot exist.

$F_8(t) = 0$ can have four double roots in the following cases:

(1). Four planes with five-point contact. The equations could be written in the form

$$x = t^5, \ y = (t - 1)^5, \ z = (t - k)^5 \ w = 1.$$

(2). Two five-point planes and two inflexions.
(3). One five-point plane and three inflexions.
(4). Four inflexions.
(5). Four inflexions and one bitangent.

The curve with cusps at 0 and ∞ has for its equations, $x = t^4(t+a)$, $y = t^3$, $z = t^2$, $w = t + d$. By applying the analytic test it may be found that it is impossible that this curve should belong to a linear complex. Hence, there is no rational quintic having two cusps and an inflexion.

$$F(t) = 4t^5 + (15d + a)t^4 + 4adt^3 = 0.$$

The curve $x = t^5$, $y = t^3$, $z = t^2$, $w = 1$ has two cusps, the osculating plane at each having five-point contact. It belongs to the complex $P_{14} = 5P_{23}$.

If R_5 has a cusp at a plane having five-point contact, and one inflexion its equations may be written

$$x = t^5,\ y = t^3(t+a),\ z = t^2,\ w = t + b.$$

This curve belongs to the linear complex $P_{14} - 2P_{23} = 0$ if $2a = 5b$. Hence, a R_5 exists having one cusp with osculating plane having five-point contact and two inflexions.

$$F(t) = 6t^6 + (4a + 20b)t^5 + 15abt^4 = 0.$$

Applying the condition that $F(t) = 0$ should have equal roots we find that if $a = 10b$ there is a plane with five-point contact.

The equations of R_5 with a triple point with coincident tangents at 0 and a plane having five-point contact at ∞ may be written

$$x = t^5,\ y = t^4,\ z = t^3,\ w = 1. \quad F(t) = t^6 = 0.$$

This curve cannot belong to a complex. It is impossible to set up the equations of R_5 with a triple point of this kind and a linear inflexion.

§ 8. *The Rational Quintic of the Second Kind.*

The rational quintic of the second species is the partial intersection of a quadric surface F_2 with a ruled surface of the fourth order having a quadrisecant of R_5 for triple line. The rational quintic of the second kind has ∞' quadrisecants. All these lines form one system of generators of F_2 the only quadric surface on which the curve lies. The generators forming the other system are simple secants of R_5^0. Our quintic is of type (1.4).

The curve R_5^0 can be depicted on a conic C_2. The plane of C_2 will cut F_2 in another conic C_2'; these conics intersect in four points, through each of which passes one quadrisecant, S_4. The pencil of planes through any one of these four quadrisecants will cut R_5 in one other point, and C_2 once, hence, the points, of R_5 and C_2 are in (1 . 1) correspondence. The four points on a quadrisecant will define a biquadratic involution of R_5. These points will depict into a biquadratic involution on C_2. The lines joining corresponding points of a tetrad will envelop a curve of class three, I_3 and order six. A tetrad on C_2 and any other point correspond to five points lying in a plane. If the involution curve I_3 is supposed given one can easily find the images of the three points which, with the given point t on C_2 form a tetrad by drawing the three tangents from t to I_3. These will meet C_2 in the required points t', t'', t'''.

The images t_1, t_2, t_3 of any three points of R_5 are given; to find the images of the two points t_4 and t_5 which lie in a plane with the three given points. For convenience of notation, we will consider any point t_i on C_2 as the image of (t_i) on R_5. The pencil of planes having for axis the line $(t_1)\,(t_2)$ cuts out a cubic point involution on R_5, to which the triad $(t_3) . (t_4) . (t_5)$ belongs. This will depict into a cubic involution on C_2, in which t_4, t_5 complete the triad corresponding to t_3. The plane of the pencil containing the quadrisecant through (t_1) cuts R_5 in the points (t_1'), (t_1''), (t_1''') which belong to the cubic involution. Similarly, the plane containing the quadrisecant through (t_2) cuts out the triad (t_2'), (t_2''), (t_2'''). Hence the cubic involution on C_2 is fixed by the two triads t_1', t_1'', t_1'''; t_2', t_2'', t_2'''. From t_1 draw the three tangents to I_3, which meet C_2 in the points t_1', t_1'', t_1'''. Similarly draw the tangents from t_2 to I_3, and find the points t_2', t_2'', t_2'''. The lines $t_1'\,t_1''$, $t_1''\,t_1'''$, $t_1'''\,t_1'$, $t_2'\,t_2''$, $t_2''\,t_2'''$, $t_2'''\,t_2'$, are tangent to I_3, and also touch $J_2(t_1 t_2)$, the involution curve enveloped by the lines joining corresponding points in the cubic involution. These six tangents determine a Brianchon point 0_{12}. Hence, if t_3 is given draw the two tangents from t_3 to $J_2(t_1 t_2)$; their points of intersection with C_2 are the required points t_4, t_5. The line $t_4 t_5$ also touches $J_2(t_1 t_2)$.

To determine $t_4 t_5$, one could use instead of $t_1 t_2$, the cubic involutions determined by $t_2 t_3$ or $t_3 t_1$, whose involution curves are $J_2(t_2 t_3)$ and $J_2(t_3 t_1)$. The tangents drawn from t_3 to either of these conics will meet C_2 in the points t_4, t_5.

If five points t_1, t_2, t_3, t_4, t_5 are the images of five points lying in a plane the cubic involution is determined by any one of the ten pairs of points $t_i t_k$. These determine ten corresponding involution conics $J_2(t_i t_k)$. From the previous case we know that $J_2(t_i t_j)$ is touched by three tangents $t_k t_l$, $t_l t_m$, $t_m t_k$. Each conic

$J_2(t_i t_j)$ is touched by three lines joining the t's whose subscripts are different from i and j. Each line $t_i t_j$ is tangent to three different conics $J_2(t_p t_q)$, where p and q are different from i and j. Corresponding to the ten conics there are ten Brianchon points 0_{ij}.

We have seen that corresponding to the point t_1, there are three lines $t_1' t_1''$, $t_1'' t_1'''$, $t_1''' t_1'$, which are tangent to I_3 and $J_2(t_1 t_2)$. These lines are also tangent to $J_2(t_1 t_3)$, $J_2(t_1 t_4)$, $J_2(t_1 t_5)$. The scheme may be expressed thus:

$$t_1' t_1'',\ t_1'' t_1''',\ t_1''' t_1', \text{ touch } J_2(t_1 t_2),\ J_2(t_1 t_3),\ J_2(t_1 t_4),\ J_2(t_1 t_5),$$

and similar expressions for the others. Thus we see that of these fifteen lines, tangent to I_3, each set $t_i' t_i''$, $t_i'' t_i'''$, $t_i''' t_i'$, touches four involution conics $J_2(t_i t_k)$ where k has the four values different from i. We can summarize our results as follows:

Five points t_1, t_2, t_3, t_4, t_5 *are the images of five coplanar points of* R_5 *if the three lines joining any three of them in pairs touch the involution conic* $J_2(t_i t_k)$ *determined by the triads on the quadrisecants through the other two.*

Or:

Five points t_1, t_2, t_3, t_4, t_5 *are images of five coplanar points of* R_5 *when the line joining any two of them is tangent to each of three involution conics determined by the three pairs of triads on the quadrisecants through the other three points.*

Or:

Five points t_1, t_2, t_3, t_4, t_5 *are images of five coplanar points of* R_5 *when the three lines joining the points of intersection with* C_2 *of the three tangents to the involution curve* I_3 *from any one of the points* t_i *touch four involution conics* $J_2(t_i t_k)$, $k \neq i$.

If two of the points (t_1) (t_2) coincide, the cutting plane becomes a tangent plane. The line $t_1 t_2$ now becomes a tangent to C_2, and the points $t_1' t_2'$, $t_1'' t_2''$, $t_1''' t_2'''$ respectively coincide. The conic $J_2(t_1 t_2)$ now touches the sides of the triad $t_1' t_1'' t_1'''$ at their points of contact with I_3. Hence, *three points* (t_3) (t_4) (t_5) *lie in a plane with the tangent at* (t_1) *when the lines joining their images all touch the involution conic* $J_2(t_1, t_1)$, *which touches the sides of the triad belonging to* t_1 *at their points of contact with* I_3.

If (t_3) and (t_4) also coincide the tangent at (t_3) cuts the tangent at (t_1), and the plane becomes a double tangent plane. The line joining t_3 to t_5 is tangent to $J_2(t_1 t_1)$, t_5 being the point of contact. Similarly, $t_1 t_5$ touches $J_2(t_3 t_3)$ the point of contact being t_5 also. $J_2(t_1 t_3)$ is touched by $t_1 t_3$, $t_3 t_5$, $t_5 t_1$. The conic $J_2(t_3 t_5)$ is touched by $t_1 t_3$, the point of contact being t_1. Similarly other conics J_2 are touched by certain lines. Since t_3 and t_4 coincide, the tangent at t_3 touches $J_2(t_1 t_1)$. Hence, *to find the image of the point of contact of a tangent which touches* R_5 *in such a point that the tangent at that point cuts the tangent at* $t_1(t_1)$, *construct the conic* $J_2(t_1 t_1)$. *The common tangents to* $J_2(t_1 t_1)$ *and* C_2 *will touch* C_2 *in the required points.* There are four common tangents; therefore the given tangent is cut by four others, hence the *osculating developable is of order eight.*

Since t_3 and t_4 coincide, t_5 becomes the point of contact of the tangent from t_3 to $J_2(t_1 t_1)$. But t_5 must lie on C_2. Hence to find the image t_5 of the residual point of contact of the double tangent plane at (t_1) and (t_3) find the point of intersection t_5 of $J_2(t_1 t_1)$ with C_2 such that $t_5 t_3$ is tangent to $J_2(t_1 t_1)$. The four points of intersection are the images of the four residual points corresponding to the points of contact of the tangents which meet the tangent at (t_1). Hence, *to find the image of the second point of contact of the double tangent plane through* t_1, *construct the involution conic* $J_2(t_1 t_1)$; *any point of intersection of* $J_2(t_1 t_1)$ *with* C_2 *will be the image of* (t_5), *the residual point in the double tangent plane; the intersection of the tangent to* $J_2(t_1 t_1)$ *at* t_5 *with* C_2 *will be the required point* t_3. Since there are four points t_5 we have another proof that every tangent of R_5 is met by four others, or that through each point of the curve R_5 pass four planes, tangent in the same point and in another point of the curve.

If the image of t_3 is given instead of that of t_1 we have seen that $t_5 t_1$ is tangent to $J_2(t_3 t_3)$ at the point t_5. The points t_5 are fixed points on C_2, so that the lines $t_1 t_5$ must be tangent to $J_2(t_3 t_3)$ at the points t_5, and the points t_1 are the points of intersection of these tangents with C_2. Hence, *if* t_1, t_3, *are the images of the two points of contact of a double tangent plane, the four bitangent planes tangent to* R_5 *in* (t_1) *and in another point of the curve have the same residual points as the four bitangent planes tangent to* R_5 *in* t_3 *and in another point of the curve. The involution conics* $J_2(t_1 t_1)$, $J_2(t_3 t_3)$ *intersect* C_2 *in the same four points* t_5; *the lines* $t_5 t_3$, $t_5 t_1$ *are tangent to* $J_2(t_1 t_1)$, $J_2(t_3 t_3)$, *respectively.*

Between the points t_1 and t_3 on the carrier C_2 there is a symmetric (4, 4) correspondence with eight coincidences. At these coincidences the two points of contact of the tangent plane coincide and the plane becomes a stationary tangent plane. Hence R_5 *has eight stationary planes.* The lines $t_1 t_3$ envelop the direction curve D_4 of class four and order twelve. The points of contact of the carrier C_2 with the eight tangents common to D_4 and C_2 are the images of the points of contact of the eight stationary planes of R_5; for in each of these points, t_1 coincides with its corresponding point t_3. The twenty-four branch points in this (4, 4) correspondence are the points of intersection of D_4 with C_2. For two coincident points correspond to each branch point. Hence, two tangents come together in the branch point, or each branch point is on D_4.

If $t_1 = t_2 = t_3$ then the plane becomes an osculating plane to R_5 at (t_1). Hence, *to find the images of the remaining points of intersection of* R_5 *with the osculating plane at* (t_1), *find the conic* $J_2(t_1 t_1)$, *and from* t_1 *draw the two tangents to* $J_2(t_1 t_1)$; *they will cut* C_2 *in the coplanar points* $t_4 t_5$.

To determine how many osculating planes of R_5 can be drawn through any point of the curve, and osculate the curve elsewhere, suppose the point t_5 given. Lines from t_5 are required such that the lines $t_5 t_1$ touch the conic $J_2(t_1 t_1)$ fixed by the other point of intersection with C_2. Corresponding to each point t_1 is only one conic $J_2(t_1 t_1)$. How many conics $J(t, t)$ are touched by an arbitrary secant $t_5 t_1$? Let the tangent at (t) cut (t_5) (t_1). The cubic involution determined by the pencil of planes through the tangent at (t) will determine the conic $J_2(tt)$. In any plane of the pencil there are three other points (t). The lines joining the images of these points touch $J_2(tt)$ Hence $t_5 t_1$ will touch $J_2(tt)$ if (t_5) (t_1) is in a plane with the tangent at (t). Hence $t_5 t_1$ touches as many conics $J_2(tt)$ as there are tangent lines that cut the chord (t_5) (t_1), that is eight. There is a $(1 . 8)$ correspondence between the points (t_1) and the lines (t_5) (t_1). A line (t_5) (t_1) determines one point (t_1) and eight conics $J_2(t, t)$. The point (t_1) determines one conic $J_2(t . t)$. As t_1 describes C_2 there may be nine coincidences of $J_2(t_1 t_1)$ with one of the eight conics $J_2(tt)$. When $(t_1) = (t_5)$ the plane becomes an osculating plane and only four other tangents cut $(t_5)(t_5)$; that is, three have disappeared, hence $t_1 = t_5$ is a triple self-corresponding element when $t = t_1$. Therefore, *from any point of* R_5 *there can be drawn six planes which osculate the curve elsewhere. The osculating developable is of class nine.*

The biquadratic involution on C_2 defined by the quadrisecants of R_5 has six double points d_1, d_2, d_3, d_4, d_5, d_6, and twelve corresponding branch points. The double points are the points of contact of the six tangents common to I_3 and C_2; the branch points are the points of intersections of the two curves. The six double points are the images of the points of contact of the six quadrisecants which are tangent to R_5. The branch points are the remaining points of intersection of these six quadrisecants with the curve.

The characteristic numbers for the (1 . 4) unicursal R_5 are in the general case:

$$\begin{aligned} n &= 5. & g &= 20. \\ r &= 8. & x &= 16. \\ m &= 9. & y &= 12. \\ h &= 6. & \alpha &= \ \ 8. \end{aligned}$$

An exhaustive classification would include all the possible configurations of the nodal curve and the double developable. The maximum genus of x is 3; if it break up into two curves having tangents of R_5 for bisecants of each, both must be unicursal. If one irreducible component has the tangents for trisecants its maximum genus is 1.

The equations of this quintic can be written in terms of the parameters of the points on two quadrisecants, and on two secants from the other system of generators. By means of a suitable transformation two points on one quadrisecant may be taken as 0, and 1, and one point on the other quadrisecant as ∞. Denoting the parameters of the points on the two secants by m and p, we can write the equations.

$$\begin{aligned} x &= t(t-1)\,(t-n)\,f_2(t) \\ y &= \quad (t-n)f_3(t). \\ z &= t(t-1)\,(t-p)\,.\,f_2(t). \\ w &= \quad (t-p)\,.\,f_3(t). \end{aligned}$$

where, x, y, z, w are the four planes passing through the consecutive sides of the skew quadrilateral, forming the tetrahedron of reference. $\frac{x}{z} = \frac{y}{w} = \frac{t-n}{t-p}$. The curve lies on the hyperboloid $xw = yz$.

We will now consider the various types of (1 . 4) rational quintics. Since $h = 6 = \frac{4\,.\,3}{2}$, there can be no cusps or actual double points. The possible line singularities are bitangents, stationary tangents, or tangents with four-point contact.

1. *One Bitangent.*

The biquadratic involution on C_2 is now determined by one group $t_1 t_2 t_3 t_4$ and a group consisting of two double elements $d_{12} d_{34}$. I_3 is touched by the six sides of the complete quadrilateral $t_1 t_2 t_3 t_4$, the line $d_{12} d_{34}$ is double tangent to I_3, its points of contact forming with $d_{12} d_{34}$, a harmonic range. The tangents to C_2 at $d_{12} d_{34}$ touch I_3; and I_3 can be considered as the envelope of the tangents drawn at the intersections of $d_{12}d_{34}$ with the conics circumscribed to the quadrilateral $t_1t_2t_3t_4$. This quadrilateral can be replaced by any other tetrad of the involu tion. Hence the points of contact of the line $d_{12}d_{34}$ with I_3 are points on the two conics which touch $d_{12}d_{34}$ and are circumscribed to any quadrilateral whose vertices represent a tetrad of the biquadratic involution. Every curve of class three with a double tangent can enter as involution curve for ∞' biquadratic involutions. It is only necessary to circumscribe a conic to any quadrilateral circumscribed to I_3. We have, $x = 15$, $y = 11$, $\omega = 1$.

The equations are:

$$\begin{aligned} x &= t^2(t-1)^2(t-n), \\ y &= \quad (t-n)f_3(t), \\ z &= t^2(t-1)^2(t-p), \\ w &= \quad (t-p)f_3(t), \end{aligned}$$

and $xw = yz$ is the equation of the hyperboloid on which R_5 lies.

2. *Two Bitangents.*

The involution on C_2 is now determined by two groups consisting of double elements. Let the elements of these groups be $d_{12}d_{34}$, $d'_{12}d'_{34}$. The involution curve of class three has now two double tangents and therefore, breaks up into a conic J_2, and J_1, the point of intersection of $d_{12}d_{34}$ and $d'_{12}d'_{34}$, the two double tangents. J_2 must touch the two double tangents and the tangents to C_2 at the points d_{12}, d_{34}, d'_{12}, d'_{34}. The points of contact of this conic with these double tangents are harmonic conjugates to J_1 with respect to $d_{12}d_{34}$ and $d'_{12}d'_{34}$, respectively, for the same reasons as in the case for one bitangent. The line I joining these points is therefore the polar of J_1 with respect to J_2 and C_2. Hence I is a side and J_1 the opposite vertex of the self-conjugate triangle common to both conics. If the two tangents are drawn from J_1 to C_2, their points of contact e_{13}, f_{13} represent the two remaining double elements of the involution. Since e_{13} and f_{13} lie on the polar of J_1 we see that they separate harmonically each pair $d_{12}\,d_{34}$, $d'_{12}\,d'_{34}$. Hence, *if a biquadratic involution contains two groups consisting only of double elements, the harmonic element pair common to both these groups represents the fifth and the sixth double element of the involution.*

The Cayley numbers are:

$$x = 14, \quad y = 10, \quad \omega = 2.$$

The equations are obtained from last case by replacing $f_3(t)$ by $(t-k)^2$.

3. *Three Bitangents.*

The biquadratic point involution on C_2 now contains three groups consisting only of double elements. The involution is fixed by two such groups. In the previous case it was shown that the two remaining double elements are the points of contact of the tangents drawn from J_1 to C_2. Let the lines joining $d_{12}\, d_{34}$, $d'_{12}\, d'_{34}$ be D, D' respectively, and the tangents to C_2 at these points D_{12}, D_{34}, D'_{12}, D'_{34}. The lines D, D' can be chosen as conjugate polars of C_2. Then $D_{12}\, D_{34}$ meet on D' and $D'_{12}\, D'_{34}$ meet on D. J_2 breaks up into the two points which lie on the polar of J_1 with respect to C_2 and with J_1 form the vertices of a self-conjugate triangle with respect to C_2. Hence, *if a biquadratic point involution on C_2 possesses three groups consisting only of double elements the involution curve I_3 breaks up into three points which are the vertices of a self-conjugate triangle with respect to C_2; the tetrads form quadrilaterals inscribed to C_2 which have these points for intersections of the diagonals. Or, the ordinary tetrads of the involution consist of the vertices of the ∞^1 quadrangles inscribed in C_2 which have the self-conjugate triangle as diagonal triangle.*

The Cayley numbers for this case are:

$$x = 13, \quad y = 9, \quad \omega = 3.$$

Since there are three double tangents there must be a specialisation in the formulas. Taking 0, 1, and ∞ as three roots the involution can be written in the form

$$t^2 (t-1)^2 = m\,(t-k)^2.$$

If this has a pair of equal roots for one group there must be some values of m and k for which this equation is a perfect square. These give $m = -1$, $k = 1/2$. The parameters of the points of tangency of the two bitangents are 0, 1, and ∞, $1/2$. The equations are obtained from the last case by putting $k = 1/2$. Since the equations determining m and k are linear the curve cannot have more than three bitangents. This is also evident from the fact that each bitangent requires two of the six double elements of the biquadratic involution on C_2; moreover it would be impossible to have an involution curve of class three with four double tangents.

4. *One Linear Inflexion.*

One group of the biquadratic involution on C_2 consists of the points $t_{123}\, t_4$. Three tangents can be drawn to I_3 from t_{123}, hence t_4 is a cusp on the involution curve I_3 and $t_{123}\,t_4$ is the cuspidal tangent. The three tangents $t_1\,t_2$, $t_2\,t_3$, $t_3\,t_1$, coincide in the tangent to C_2 at t_{123}. Hence I_3 touches C_2 at the point t_{123}. Hence we see that a triple element in the biquadratic involution absorbs two double elements.

The Cayley numbers are:

$$\begin{array}{ll} n=5 & g=15 \\ h=6 & x=15 \\ v=1 & y=12 \\ r=8 & \alpha=6 \\ m=8 & \end{array}$$

The equations are:

$$\begin{aligned} x &= t^3\,(t-1)\,(t-n) \\ y &= (t-n)\,.\,f_3(t)\,. \\ z &= t^3\,(t-1)\,(t-p)\,. \\ w &= (t-p)\,.\,f_3(t)\,. \end{aligned}$$

5. *One Linear Inflexion and one Bitangent.*

The involution on C_2 is now given by the two groups $d_{12}\,d_{34}$, $t_{123}\,t_4$. From the previous treatment we know that I_3 has the line $d_{12}\,d_{34}$ for double tangent and that its points of contact with I_3 form with $d_{12}\,d_{34}$ a harmonic range. I_3 has a cusp at t_4 and touches C_2 at t_{123}. It also touches the tangents to C_2 at $d_{12}\,d_{34}$. This case is possible for I_3 may have both a cusp and a double tangent.

The Cayley numbers become:

$$\begin{array}{ll} n=5. & m=8. \\ h=6. & g=15. \\ v=1. & x=14. \\ \omega=1. & y=11. \\ r=8. & \alpha=6. \end{array}$$

The equations may be written:

$$\begin{aligned} x &= t^3\,(t-1)\,(t-n). \\ y &= (t-k)^2\,(t-n). \\ z &= t^3\,(t-1)\,(t-p). \\ w &= (t-k)^2\,(t-p). \end{aligned}$$

6. *Two Linear Inflexions.*

The involution on C_2 contains two groups $t_{123}t_4$, $t'_{123}t'_4$. The involution curve I_3 has cusps at t_4, t'_4. The lines $t_{123}t_4$, $t'_{123}t'_4$ are cuspidal tangents, and I_3 will touch C_2 at t_{123}, t'_{123}.

$$\begin{array}{ll} n = 5. & g = 11. \\ h = 6. & x = 14. \\ v = 2. & y = 12. \\ r = 8. & \alpha = \ \ 4. \\ m = 7. & \end{array}$$

The equations of this curve are ;

$$\begin{aligned} x &= t^3(t-1)\,(t-n). \\ y &= \quad (t-k)\,(t-n). \\ z &= t^3(t-1)\,(t-p). \\ w &= \quad (t-k)\,(t-p). \end{aligned}$$

7. *Two Linear Inflexions and One Bitangent.*

The involution on C_2 has three groups of the form $t_{123}t_4$, $t'_{123}t'_4$, and $d_{12}d_{34}$. I_3 has $d_{12}d_{34}$ for double tangent, and the lines $t_{123}t_4$, $t'_{123}t'_4$ for cuspidal tangents. I_3 touches C_2 at the points $t_{123}t'_{123}$: t_4 and t'_4 are cusps on I_3. Since there is one double tangent the order of I_3 reduces to four, and the number of cusps reduces to three.

The Cayley numbers are:

$$\begin{array}{ll} n = 5. & m = \ \ 7. \\ h = 6. & g = 11. \\ v = 2. & x = 13. \\ \omega = 1. & y = 11. \\ r = 8. & \alpha = \ \ 4. \end{array}$$

Since there are two triple elements the involution can be written $t^3(t-1) = m(t-k)$. If, in addition there is one group consisting of double elements, this must be a square for some value of m and k.

We have $m = k = -1/8$. Hence the equations for the curve become:

$$\begin{aligned} x &= t^3(t-1)\,(t-n)\,. \\ y &= (t+1/8)\,(t-n)\,. \\ z &= t^3(t-1)\,(t-p)\,. \\ w &= (t+1/8)\,(t-p)\,. \end{aligned}$$

Since this method gave just one value for m and k it follows that the R_5 with two linear inflexions cannot have more than one bitangent. This can also be shown by means of the involution curve.

8. *Three Linear Inflexions.*

The point involution on C_2 now contains three groups $t_{123}\,t_4$, $u_{123}\,u_4$, and $v_{123}\,v_4$. I_3 has three cusps at t_4, u_4, and v_4; the lines $t_{123}\,t_4$, $u_{123}\,u_4$, $v_{123}\,v_4$ are the cuspidal tangents and must therefore meet in a point. I_3 also touches C_2 at t_{123}, u_{123}, v_{123}.

The Cayley numbers become:

$$\begin{array}{ll} n = 5\,. & g = 8\,. \\ h = 6\,. & x = 13\,. \\ v = 3\,. & y = 12\,. \\ r = 8\,. & \alpha = 2\,. \\ m = 6\,. & \end{array}$$

The involution is given by the equation $t^3\,(t-1) = m\,(t-k)$. When the three roots are equal, $mk - \frac{m}{4} = 0$, and $\frac{m^2}{16} + \frac{mk}{16} = 0$. Therefore $m = -1/4$, $k = 1/4$. The equations may be written:

$$\begin{aligned} x &= t^3\,(t-1)\,(t-n)\,. \\ y &= (t-1/4)\,(t-n)\,. \\ z &= t^3\,(t-1)\,(t-p)\,. \\ w &= (t-1/4)\,(t-p)\,. \end{aligned}$$

The involution has three triple elements for which $t = 0$, $t = \infty$, and $t = 1/2$. This also shows that the involution cannot have more than three triple elements. Hence, the R_5 cannot have more than three linear inflexions. This is also evident geometrically. For C_2 and I_3 in the case of three linear inflexions touch in three points and three cusps of I_3 are on C_2. Hence the curves meet in twelve points. An additional bitangent or inflexion would require additional points of intersection of C_2 and I_3, which is impossible.

9. *One Tangent of Four-point Contact.*

If the points $t_{12}\,t_{34}$ in the case when one group is composed of double elements become coincident we have one group consisting of a fourfold element. The tangent to C_2 at t_{1234} is a double tangent of I_3 and t_{1234} is one of the points of contact. The other point of contact of the double tangent is the point where it touches the second conic circumscribed to any quadrilateral $t_1 t_2 t_3 t_4$ and tangent to the double tangent. The tangent at t_{1234} counts for three common tangents to C_2 and I_3. Hence, a four-fold point on R_5 absorbs only three of the double elements of the involution.

The number of stationary planes on R_5 reduces to four, and the order of the nodal curve to thirteen; hence, for this case we have $v = 2$, $\omega = 1$, $n = 5$, $h = 6$, $r = 8$, $x = 13$, $m = 7$, $g = 11$, $\alpha = 4$, $y = 11$.

The equations are:

$$\begin{array}{ll} x = t^4 (t - n) . & y = (t - n) \cdot f_3 (t) . \\ z = t^4 (t - p) . & w = (t - p) \cdot f_3 (t) . \end{array}$$

10. *One Four-point Contact Tangent and one Bitangent.*

In the biquadratic point involution considered on p. 330, the points $d_{12} d_{34}$ coincide. The involution curve consists of the point J_1 and the conic J_2. J_2 touches C_2 at the point d_{1234}; it also touches the tangents to C_2 at the points $d'_{12} d'_{34}$, and the line $d'_{12} d'_{34}$.

The Cayley numbers are:

$$\begin{array}{ll} n = 5 . & m = 7 . \\ h = 6 . & g = 11 . \\ v = 2 . & x = 12 . \\ \omega = 2 . & y = 10 . \\ r = 8 . & \alpha = 4 . \end{array}$$

The equations are:

$$\begin{array}{ll} x = t^4 (t - n) . & y = (t - k)^2 (t - n) . \\ z = t^4 (t - p) . & w = (t - k)^2 (t - p) . \end{array}$$

11. *One Four-point Contact Tangent and one Linear Inflexion.*

The point involution on C_2 has two groups d_{1234} and $t_{123} t_4$. The tangent to C_2 at d_{1234} is a double tangent to I_3, and I_3 touches C_2 at d_{1234}. I_3 has a cusp at t_4; $t_{123} t_4$ is the cuspidal tangent; and C_2 touches I_3 at t_{123}.

The Cayley numbers are:

$$\begin{array}{ll} n = 5 . & m = 6 . \\ h = 6 . & g = 8 . \\ v = 3 . & x = 12 . \\ \omega = 1 . & y = 11 . \\ r = 8 . & \alpha = 2 . \end{array}$$

The equations are :

$$\begin{array}{ll} x = t^4 (t - n) . & y = (t - k) (t - n) . \\ z = t^4 (t - p) . & w = (t - k) (t - p) . \end{array}$$

12. *Two Four-point Contact Tangents.*

The involution on C_2 contains the groups $d_{1234}\, d'_{1234}$. The tangents to C_2 at these points meet in the point J_1. J_2 will touch C_2 at the points $d_{1234}\, d'_{1234}$.

The Cayley numbers are :

$$\begin{array}{lll} n = 5 . & r = 8 . & y = 10 . \\ h = 6 . & m = 5 . & \alpha = 0 . \\ v = 4 . & g = 6 . & \\ \omega = 2 . & x = 10 . & \end{array}$$

The equations are :

$$\begin{array}{ll} x = t^5 & y = t \\ z = t^4 & w = 1 . \end{array}$$

Two bitangents and one linear inflexion is an impossible case. Consider the involution on C_2 determined by two groups consisting of double elements. It has been shown that I_3 breaks up into a point J_1 and a conic J_2. A triple element t_{123} requires a cusp on I_3 at the point t_4, which is impossible in this case. This can also be shown analytically. The involution determined by the two bitangents is of the form

$$t^2 (t - 1)^2 = m (t - k)^2.$$

If the curve has in addition one linear inflexion this equation must have a triple root for some suitable values of m and k. But this requires conditions which are incompatible and the additional inflexion is impossible.

§ 9. *Curves that belong to a Linear Complex.*

(1) Four linear inflexions.

$$x = t^4 \left(t - \frac{25}{7} c\right), \quad y = t^3 (t - 5c), \quad z = t + c, \quad w = t\left(t + \frac{5c}{7}\right).$$

This curve belongs to the complex $P_{13} + 2P_{42} = 0$.

(2) One four-point inflexion, two linear inflexions. It belongs to the complex $P_{13} + 2P_{42} = 0$.

$$x = t^5, \quad y = t^4, \quad z = t + c, \quad w = t\left(t + \frac{5c}{6}\right).$$

(3) Two cusps at planes with five-point contact.

$$x = t^5, \quad y = t^3, \quad z = t^2, \quad w = 1.$$

This is a (2, 3) curve on the hyperboloid $xw = yz$, and belongs to the complex $P_{14} - 5P_{42} = 0$. The developable is of order six and the nodal curve is a rational R_5 having two double points.

(4) Two four-point inflexions.

$$x = t^5, \quad y = t^4, \quad z = t, \quad w = 1.$$

This is a (1, 4) curve on the hyperboloid $xw = yz$. It belongs to the complex $3p_{14} - 5p_{23} = 0$.

(5). Two linear inflexions and one cusp at a plane with five-point contact.

$$x = t^5, \quad y = t^3 (t + a), \quad z = t^2, \quad w = t + \frac{2a}{5}.$$

This quintic belongs to the complex $P_{14} - 2P_{23} = 0$.

(6) Four inflexions, one bitangent.

$$x = t^4 \left(t + \frac{b}{2}\right), \quad y = t^3 (t + b), \quad z = t + \frac{b}{4}, \quad w = t^4 \left(t + \frac{b}{8}\right).$$

The curve belongs to the complex $P_{13} + 2P_{42} = 0$.

A rational quintic curve belonging to a linear complex projects from an arbitrary point in space into a rational plane quintic curve C_5 having five inflexions on a straight line. Dually, the section of the developable is a plane curve with five cusps, and the cuspidal tangents all meet in one point, the pole of the given plane.

CHAPTER II.

ON THE QUINTIC CURVES OF GENUS ONE.

The quintic curve of genus one R'_5 projects from one of its own points in a plane quartic curve of genus one, with two double points. Hence, through each point of R_5 pass two trisecants. Any trisecant T is met by three trisecants. The order of the ruled surface formed by the trisecants is two greater. Hence, the trisecant surface F_5 of R_5 is of order five. R_5 is the only double curve of the surface. Hence, F_5 is of genus one.

Every surface of the third order through fourteen points of R_5 cuts R_5 in one and the same fifteenth point. For, pass the pencil of the surfaces through the given fourteen points and four points not on R_5. Each surface cuts R_5 in one more point. But a twisted curve of genus one cannot have a pencil of surfaces that cut it in one variable point besides the fixed points. On the contrary there is an infinite number of adjoined surfaces that cut it in variable point pairs. Hence, the surfaces through fourteen points pass through a fixed fifteenth point. All the F_3 through fourteen points do not contain R_5; for, take five points in the plane α, five points in β, and four in γ. The planes α, β, γ represent a F_3 on which R_5 does not lie.

Take any fourteen points as basis points. Pass F'_3 through these points. F_3 and F'_3 intersect in fifteen points on R_5. Hence R_5 meets F_3 in sixteen points, and R_5 lies on F_3. Hence, if a surface F_3 contains the curve R_5, this counts for fifteen simple conditions.*

A surface of the third order is determined by nineteen conditions. Hence, through R_5 and any four points of space not on R_5, a surface F_3 containing R_5 is determined. All the F_3 through R_5 and three points form a system of ∞' surfaces. The surfaces F_3 passing through R_5 form a linear system of ∞^4 surfaces. A line counts for four conditions. Hence through R_5 and any line g of space can be drawn one cubic surface.

Let F_3 be any cubic surface through R_5. The trisecant surface F_5 meets F_3 in a curve of order fifteen, in which R_5 is a double curve. Let A be any point on the residual curve. The trisecant a_1 through A meets R_5 and therefore F_3 in three other points. Hence a_1 is a line on F_3. Hence, the residual curve of order five is composed of five skew lines, a_1, a_2, a_3, a_4, a_5.

Let g and g' be the two transversals of the lines a_1, a_2, a_3, a_4; they meet F_5 in five points each, and are lines on the surface F_3. The fifth point of intersection of each line with F_5 must therefore be on the curve of intersection of F_3 and F_5. Hence, both lines g and g' cut a_5, and every quintuple a_1, a_2, a_3, a_4, a_5, is a quintuple with two transversals.

Pass an F_3 through g and R_5. g meets F_5 in five points through which pass five trisecants a_1, a_2, a_3, a_4, a_5, each of which has four points on F_3 and therefore lies on F_3. The intersection of F_3 and F_5 consists of R_5 counted twice and a_1, a_2, a_3, a_4, a_5, which have two transversals g and g'. Since g' can be found by means of g, and g by means of g', it follows that the lines of space are paired such that each pair of lines is cut by the same five trisecants, and if four trisecants a_1, a_2, a_3, a_4 are met by the two transversals g and g', then g and g' cut the same fifth trisecant a_5 and are corresponding lines. The same result will now be obtained by depicting F_3 and R_5 on a plane.

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* See also Halphen, Sur la classification des courbes gauches algébriques, *Journal de l'Ecole polytechnique.* Vol. 52, p. 41.

Take a line a_1 and five other lines b_2, b_3, b_4, b_5, b_6 which do not intersect cutting it. These lines are on a cubic surface. For take four points in which a_1 cuts say $b_2 \ldots . b_5$, and three other points on each of the lines $b_2 \ldots . b_6$. The nineteen points determine a cubic surface through these six lines. Four skew lines in space have in general two transversals, hence through b_3, b_4, b_5, b_6 we can draw another transversal a_2 meeting F_4 in four points, and hence lying on the surface. Similarly we can find the lines a_3, a_4, a_5, a_6. Draw a line cutting the lines a_2, a_3, a_4, a_5; it will also cut a_6. Call the line b_1. These lines a_i, b_i form a *double six.* The remaining fifteen lines lie in the planes determined by a_i and b_k $(i \neq k)$. The lines $a_i b_k$ and $a_k b_i$ determine the same line r_{ik}. It can now be seen that any two b_i cut four a_k $(k \neq i)$, *e. g.*, b_2 and b_3 cut a_1, a_4, a_5, a_6; b_1 and b_6 cut a_2, a_3, a_4, a_5. b_1 also cuts r_{12}, r_{13}, r_{14}, r_{15}, r_{16}; b_6 cuts r_{61}, r_{62}, r_{63}, r_{64}, r_{65}. Hence a_2, a_3, a_4, a_5, and r_{16} cut both b_1 and b_6.*

Now depict the surface on a plane by means of the lines b_1 and b_6. The lines a_2, a_3, a_4, a_5, r_{16} meet both b_1 and b_6 and hence project into points. The other five lines meeting b_1 are a_6 and r_{12}, r_{13}, r_{14}, r_{15}; these project into the lines of a pencil through b_1. Similarly the lines a_1, r_{62}, r_{63}, r_{64}, r_{65} project into the lines of a pencil through b_6. Let b_1 and b_6 cut the plane in the points b_1 and b_6. a_1 and a_6 intersect in r_{16}; a_2 is cut by r_{12} and r_{62}, hence a_2 projects into the intersection of r_{12} and r_{62}; similarly for a_3, a_4, a_5. There are ten other lines r_{23}, r_{24}, r_{25}, r_{34}, r_{35}, r_{45}, b_2, b_3, b_4, b_5 which do not cut either b_1 or b_6. They depict into conics. The line r_{23} cuts a_2, a_3, b_2, b_3, r_{14}, r_{56}, r_{15}, r_{46}, r_{16}, r_{45}; its image is a conic through the five points a_2, a_3, r_{16}, b_1, b_6. The line b_2 cuts a_i $(i \neq 2)$ and r_{i2} $(i \neq 2)$; hence it depicts into a conic through the points a_3, a_4, a_5, b_1, b_6. The complete depiction of the twenty-seven lines consists of:

(1) two points b_1, b_6.

(2) five points a_2, a_3, a_4, a_5, r_{16}.

(3) five lines a_6, r_{12}, r_{13}, r_{14}, r_{15} joining b_1 to the points r_{16}, a_2, ...

(4) five lines a_1, r_{62}, r_{63}, r_{64}, r_{65} joining b_6 to the points r_{16}, a_2, ...

(5) ten conics r_{23}, r_{24}, r_{25}, r_{34}, r_{35}, r_{45}, b_2, b_3, b_4, b_5, passing through b_1, b_6, and three of the points $a_2 \ldots . r_{16}$.

Suppose a $R_5 (p = 1)$ passed through three points on a_2, a_3, a_4, a_5, r_{16}. From any point of R_5 one line can be drawn cutting b_1 and b_6. These lines generate a ruled surface of order ten on which b_1 and b_6 are five-fold lines, and the trisecants a_2, a_3, a_4, a_5, r_{16} are triple generators; there are no other nodal lines. The section of the surface by the plane of depiction is the image of R_5. This is a plane curve C_{10} of order ten having five-fold points at b_1 and b_6 and triple points at a_2, a_3, a_4, a_5, r_{16} and no other double points. Its genus is $P = \frac{9 \,.\, 8}{2} - 2 \,.\, \frac{5 \,.\, 4}{2} - 5 \,.\, \frac{3 \,.\, 2}{2} = 1$.

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* Salmon, Analytical Geometry of Three Dimensions.

The plane of depiction cuts F_3 in a cubic curve C_3 which passes simply through the seven fundamental points, and cuts every other a, b, r in one point apart from a fundamental point.

Any line, say a_6, that meets b_1, but not b_6, goes into a line cutting C_{10} in ten points, of which five are at b_1, three at r_{16}, and two others are simple points. Hence, a_6, and similarly r_{12}, r_{13}, r_{14}, r_{15}, a_1, r_{62}, r_{63}, r_{64}, r_{65} are bisecants of the original curve R_5. The line r_{23}, not cutting b_1 or b_6 goes into a conic C_2 through b_1, b_6, a_2, a_3, r_{16}, cutting C_{10} in twenty points. There is just one point of intersection apart from the fundamental points. Hence r_{23}, r_{24}, r_{25}, r_{34}, r_{35}, r_{45}, b_2, b_3, b_4, and b_5 are unisecants of R_5. The lines of F_3 meet R_5 as follows:

(1) a_2, a_3, a_4, a_5, r_{16}, are trisecants.

(2) a_6, r_{12}, r_{13}, r_{14}, r_{15}, a_1, r_{62}, r_{63}, r_{64}, r_{65} are bisecants.

(3) r_{23}, r_{24}, r_{25}, r_{34}, r_{35}, r_{45}, b_2, b_3, b_4, b_5, are unisecants.*

(4) b_1, b_6, do not meet R_5.

From this it follows that the lines of space are in pairs a, a', for the surface through fifteen points on R_5 and four points on b_1 also contains b_6, and similarly the surface through R_5 and b_6 contains b_1. Hence every line of space belongs to such a pair and with R_5 determines a surface F_3.

Let g go through a point A of R_5 in which the trisecants a_1, a_2 meet. Then the three trisecants a_3, a_4, a_5 through the three further points of intersection of g and F_5 form with a_1 and a_2 a quintuple. The line of intersection g' of the planes containing a_1, a_2, and a_3, a_4, meets four lines of the quintuple and therefore corresponds to g. Hence; to the lines g of the bundle with vertex at a point A on R_5 correspond the lines of the plane α containing the two trisecants a_1, a_2 through the point A.

If g lies in the plane α, g', coincides with g; hence, the lines of the plane pencil with vertex at a point A on R_5, and lying in the plane α of the trisecants a, a' through A are self-corresponding. From this, it follows that every trisecant is paired with itself.

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* D. Montesano in "Su la curva gobba di 5° ordine e di genere 1"; *Rendiconte dell 'Accademia di Napoli delle scienze fisische e matematiche,* 2nd series, vol. II, 1887, p. 182, states incorrectly that of the twenty-seven lines of F_3, twenty are bisecants.

Next, suppose g to cut R_5 in the points A, B, through which pass the trisecants a_1, a_2, a_3, a_4. The plane α containing a_1, a_2, and the plane β containing a_3, a_4 intersect in the line g' corresponding to g. Hence, to the line joining two points A, B of R_5 corresponds the line of intersection of the planes α, β containing the trisecants at the two points. Since g and g' are both met by the same fifth trisecant of the quintuple, it follows that the lines AB and $(\alpha\beta)$ are met by the same trisecant.

The surface F_3 containing a pair of corresponding lines a, a', and a surface F_3' containing a pair b, b' intersect in R_5' and a curve R_4^0 of order four and genus zero. Since a and a' meet F_3' in three points they are trisecants of R_4; similarly, it follows that b, and b' are also trisecants. The lines a, a', b, b', therefore belong to a fixed hyperboloid, since R_4^0 is a (1, 3) curve on one hyperboloid.

Hence, by the curve R_5' is determined a linear complex so that any point of R_5 corresponds to the plane determined by the trisecants through that point, and the line joining any two points corresponds to the intersection of the two corresponding planes; since the trisecants are self-corresponding lines in the system, they belong to a linear complex.

Let a pair of conjugate lines a, a' approach coincidence and let F_3 be the surface of the system containing a. Then a is a line of the complex. Each plane through a has two points of contact on a; hence, there is a (1, 2) correspondence between the planes through a and the points of a. Each plane through a has its pole on a; hence, there is a (1.1) relation between the planes through a and their poles on a. Two such correspondences can have three common points. But a is met by five trisecants, and therefore there are five planes such that the pole is at one point of contact. Hence, one point of contact is fixed, and the planes through a are projective to the variable point of contact. Therefore, there is a double point of F_3 on a.

Conversely, every point of space is a double point on an F_3 through R_5; there are ∞^3 such surfaces. The tangent cone at the double point P of F_3, meets F_3 in six lines. To study the lines on F_3 depict the surface on a plane α. Each line through P meets F_3 in one more point. The six lines through P depict into points a, b, c, d, e, f, The other fifteen lines of F_3 go into the lines joining the fundamental points. It follows that one line a does not meet R_5, that b, c, d, e, f are bisecants, that the five lines meeting a are trisecants, and that the other ten lines are unisecants. R_5 goes into a plane quintic curve with double points at b, c, d, e, f. Hence this method of depiction gives the simplest monoidal representation of R_5'. The two lines a, a' of the general case coincide in a; it follows that a is a line of the complex.

From what has just been proved it follows that the lines a of the complex and the points P of space are in (1 . 1) correspondence, such that a line a of the complex and R_5 determines a F_3 with a point on a for double point, and that the point P and R_5 determines a F_3 with double point at P, and containing a. Call this correspondence χ.

Every point of R_5 is on all the surfaces F_3; hence, in χ to each point of R_5 correspond all the lines of the complex through that point. The points of R_5 are therefore singular points in χ.

Consider the planes through a trisecant S_3. The poles of these planes are the points of S_3, so that the trisecant corresponds to all its points. Hence, the lines of the trisecant surface F_5 are singular lines in the correspondence χ.

Take any plane α and its pole P in the linear complex. The locus of the points that correspond in χ to the lines of a plane pencil in α with vertex at P is a conic C_2 through P, since there is one corresponding point on each line. The plane α cuts R_5 in five points A, B, C, D, E. To the lines PA, PB, PC, PD, PE correspond the points A, B, C, D, E; hence, C_2 is the conic through A, B, C, D, E. But the polar planes of the points A, B, C, D, E all go through the pole P of α. Hence; *if any plane α cuts R_5 in the five points A, B, C, D, E, the five planes determined by the pair of trisecants through each point all pass through P the pole of the plane α in the complex, and P lies on the conic determined by A, B, C, D, E.*

It has been proved by Fiedler that all self-dual curves of order five are of genus zero.*

CHAPTER III.

ON THE QUINTIC CURVES OF GENUS TWO.

The quintic curve of genus two is the partial intersection of a quadric surface F_2 with a cubic surface, the residual intersection being a trisecant of R_5. We will study it as a (2, 3) curve on a hyperboloid F_2.

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*See W. Fiedler, *Vierteljahresschrift der naturforschenden Gesellschaft in Zurich*, vol. XX, pp. 173–176.

To depict the surface on a plane E take any point A on F_2 but not on R_5.[*] Any line through A meets F_2 in one point, and the plane of depiction in one point. This gives a (1, 1) correspondence between the points of F_2 and E. The two generators through A meet E in the points O and O', hence O corresponds to all the points on OA, and O' to all the points on $O'A$. The line l joining OO' corresponds to A on F_2; the different points on l correspond to different directions through A. The fundamental elements of the correspondence are O, O' and l on E, and OA, $O'A$, and A on F_2.

It is shown by Clebsch that C_5, the plane projection of R_5 from A has for its equation.

$$\sum C_{abc}\, x^a\, y^b\, z^c = 0,$$

subject to the conditions

$$a + b \geqq 2, \text{ and } a + c \geqq 3.$$

Hence, the equation of C_5 is

$$b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^4z + b_5x^3yz + b_6x^2y^2z + b_7x^3z^2 + b_8x^2yz^2 \\ + b_9xy^2z^2 + b_{10}x^2z^3 + b_{11}xyz^3 + b_{12}y^2z^3 = 0.$$

From O eight tangents can be drawn to touch C_5 elsewhere; hence, *eight generators of the second system are tangent two R_5 and cut R_5 in another point.*

Six tangents can be drawn from O' to touch C_5; hence, *six generators of the first system are tangent to R_5.*

The curve C_5 with a double point at O and a triple point at O' is determined by eleven other points. Hence, R_5 is determined by eleven arbitrary points on F_2.

The equation of C_5 in non-homogeneous coordinates is

$$b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^4 + b_5x^3y + b_6x^2y^2 + b_7x^3 + b_8x^2y \\ + b_9xy^2 + b_{10}x^2 + b_{11}xy + b_{12}y^2 = 0.$$

The abscissas of the points in which the line $y = mx$ cuts C_5 are given by the equation

$$(b_1 + b_2m + b_3m^2)\,x^3 + (b_4 + b_5m + b_6m^2)\,x^2 + (b_7 + b_8m + b_9m^2)\,x \\ + (b_{10} + b_{11}m + b_{12}m^2) = 0.$$

The conditions that the three roots of this equation are equal are

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3}.\dagger$$

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[*] See Clebsch, Vorlesungen über Geometrie, vol. 2, p. 414.

[†] Burnside & Panton, Theory of Equations, vol. I, p. 84.

Since a_0, a_1, a_2, a_3 are all of order two in m, there cannot be more than four values of m for which all the roots are equal. One value gives the triple point at infinity. Hence, the quintic curve of genus two cannot have more than three inflexional tangents.

Any plane cuts F_2 in a conic C_2 whose image is another conic through O, and O', meeting C_5 in five other points. Hence, the conics through O and O' meet C_5 in five points which are the images of five coplanar points of R_5. The conic may break up into a line through O and a line through O'.

To the five points P_i in which a line l through any point P in the plane E meets C_5, correspond the five points in which the plane determined by l and A meets R_5. To the points on the planes of the pencil with axis AP correspond the points on the lines of the plane pencil P. If the point P is on C_5 there are only four variable points of intersection.

The projection of the tangent plane at A is OO'.

The projection of the section of R_5 by a tangent plane to F_2 is the points on a line through O and a line through O'.

Draw a pencil of planes with OA for axis. The points in which any plane of the pencil meets R_5 go into O and the three points in which the line of intersection with E cuts C_5. Similarly, the points of R_5 on the planes through $O'A$ lie on the lines of intersection of the planes with E.

All sections by planes that do no pass through A, depict into conics through O and O'.

The ten points of intersection of R_5 with any F_2 lie on ∞' F_2, and they are all cut out by a (2 . 2) curve on the same ruled surface. To these ten points correspond the intersections of C_5 with a C_4 with double points at O and O'.

To a doubly counted plane corresponds a C_4 touching C_5 in five points not in O and O'.

If one draws through these five points and O and O' all the C_4 with double points at O and O' these meet C_5 in five other points, the images of five coplanar points.